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Zero bias anomaly in the density of states of low-dimensional metals

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Abstract. We consider the effect of Coulomb interactions on the average density of states (DOS) of disordered low-dimensional metals for temperatures T and frequencies ω smaller than the inverse elastic life-time $1/\tau_0$. Using the fact that long-range Coulomb interactions in two dimensions ($2d$) generate $\ln^2$-singularities in the DOS $\nu(\omega)$ but only ln-singularities in the conductivity $\sigma(\omega)$, we can re-sum the most singular contributions to the average DOS *via* a simple gauge-transformation. If $\lim_{\omega\to 0}\sigma(\omega) > 0$, then a *metallic Coulomb gap* $\nu(\omega) \propto |\omega|/e^4$ appears in the DOS at $T = 0$ for frequencies below a certain crossover frequency Ω_2 which depends on the value of the DC conductivity $\sigma(0)$. Here, $-e$ is the charge of the electron. Naively adopting the same procedure to calculate the DOS in quasi $1d$ metals, we find $\nu(\omega) \propto (|\omega|/\Omega_1)^{1/2}\exp(-\Omega_1/|\omega|)$ at $T = 0$, where Ω_1 is some interaction-dependent frequency scale. However, we argue that in quasi $1d$ the above gauge-transformation method is on less firm grounds than in $2d$. We also discuss the behavior of the DOS at finite temperatures and give numerical results for the expected tunneling conductance that can be compared with experiments.

PACS. 71.10.Pm Fermions in reduced dimensions – 71.23.-k Electronic structure of disordered solids – 71.30.+h Metal-insulator transitions and other electronic transitions – 72.15.Rn Localization effects (Anderson or weak localization)

1 Introduction

In the early eighties, Altshuler and Aronov [1] perturbatively studied the effect of electron-electron interactions on the density of states (DOS) of low-dimensional weakly disordered interacting electronic systems. For temperatures T and frequencies ω smaller than the inverse elastic life-time $1/\tau_0$, they found that in reduced dimensions the interplay between disorder and electron-electron interactions gives rise to singular corrections to the average DOS. In two dimensions ($2d$), the long-range Coulomb interaction leads to a $\ln^2$-correction to the average DOS [1],

$$\nu(\omega) \sim \nu_0 \left[1 - \frac{r_0}{4}\ln(|\omega|\tau_0)\ln(|\omega|\tau_1)\right] \ , \ 2d \ , \ T = 0 \ , \quad (1)$$

where

$$r_0 = \frac{1}{(2\pi)^2\nu_0 D_0} = \frac{2e^2}{(2\pi)^2\sigma_0} = \frac{1}{\pi k_F \ell} \quad (2)$$

is a dimensionless measure for the resistance of the system at frequency scale $\omega \approx \tau_0^{-1}$ (where σ_0 is the Drude conductivity), and the interaction-dependent time τ_1 is given by

$$\tau_1 = \frac{1}{D_0^2\kappa^4\tau_0} = \frac{4\tau_0}{(\kappa\ell)^4} \cdot \quad (3)$$

Here, $\kappa = 2me^2$ is the Thomas-Fermi screening wavevector in two dimensions, $\nu_0 = m/2\pi$ is the DOS at the Fermi energy (per spin projection) of electrons with effective mass m and charge $-e$, $D_0 = v_F\ell/2$ is the diffusion coefficient in $2d$, and $\ell = v_F\tau_0$ is the elastic mean free path. We use units such that $\hbar = k_B = 1$. Note that for a good metal at high densities the Thomas-Fermi screening length is short compared with the elastic mean free path ($\kappa\ell \gg 1$) so that $\tau_1 \ll \tau_0$. In quasi $1d$ metallic wires (which consist of many transverse channels but permit diffusive motion only in one direction), the leading correction to the average DOS is [1]

$$\nu(\omega) \sim \nu_0 \left[1 - \sqrt{\frac{4\Omega_1}{\pi|\omega|}}\right] \ , \ \text{quasi } 1d \ , \ T = 0 \ , \quad (4)$$

where $\nu_0 = 1/(\pi v_F)$ is the DOS per spin in $1d$, and the frequency scale Ω_1 depends on the effective electron-electron interaction constant f_0,

$$\Omega_1 = \frac{f_0^2}{32\pi D_0} \cdot \quad (5)$$

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Here, $D_0 = v_F \ell$ is the (bare) diffusion coefficient in quasi $1d$.

Obviously, the correction terms in equations (1) and (4) diverge for $\omega \to 0$, so that at low frequencies these perturbative expressions cease to be valid. What is the true low-frequency behavior of the DOS of disordered metals in reduced dimensions? The answer to this question is relevant for a number of recent tunneling experiments [2–4], where a strong suppression of the tunneling conductance $G(V)$ as a function of the applied voltage has been observed (zero bias anomaly). The tunneling conductance is related to the DOS *via* $G(V) \propto \nu(\omega = eV)$, so that the experimentally observed zero bias anomaly in the tunneling conductance reflects the strong suppression of the average DOS at the Fermi energy.

In $2d$, the low-temperature behavior of the DOS of a strongly correlated disordered metal has recently been measured by Bielejec *et al.* [2]. While at higher temperatures they found a logarithmic correction, at the lowest temperatures they found a stronger-than linear suppression of the DOS, which has been interpreted in terms of a hard correlation gap. The knowledge of the low-energy behavior of the DOS of a $2d$ disordered metal with a finite conductivity might also be important to gain a better understanding of the physical mechanism that is responsible for the apparent metal-insulator transition in doped semiconductor devices [5,6]. An intensely studied quasi $1d$ system where under certain conditions the electrons propagate diffusively in only one direction are multi-wall carbon nanotubes [3,4].

Let us briefly review previous calculations of the zero bias anomaly. In $2d$, the first attempt to determine the true low-frequency asymptotics of $\nu(\omega)$ was apparently due to Finkelstein [7] who found that for frequencies exceeding the crossover frequency

$$\Omega_2 \equiv \tau_0^{-1} \exp[-1/r_0] \,, \tag{6}$$

the perturbative expression given in equation (1) can actually be exponentiated, so that

$$\nu(\omega) \approx \nu_0 \exp\left[-\frac{r_0}{4} \ln(|\omega|\tau_0) \ln(|\omega|\tau_1)\right] . \tag{7}$$

This expression has been re-derived in different ways by several authors [8–11]. It should be emphasized that equation (7) is not valid for frequencies smaller than the crossover frequency Ω_2 defined in equation (6) [7,8,10,11]. For $\omega \to 0$ Finkelstein [7] found that $\nu(\omega) \propto |\omega|^{1/4}$. However, in the derivation of this result he assumed that the conductivity $\sigma(\omega)$ diverges logarithmically for $\omega \to 0$. The behavior of the DOS of $2d$ disordered electrons with a finite conductivity which is of experimental interest was not calculated by Finkelstein. A simple interpolation formula for the DOS, which yields a physically sensible result even for $\omega \to 0$, has been proposed by one of us in reference [10]. This formula is based on a re-summation of the leading $\ln^2$-singularities to all orders in perturbation theory, consistently neglecting sub-leading terms that involve only logarithmic corrections. In this approximation, one obtains

$$\nu(\omega) \approx \nu_0 \frac{2}{\pi} \int_{\tau_0}^{\infty} \mathrm{d}t \frac{\sin(|\omega|t)}{t} \exp\left[-\frac{r_0}{4} \ln(t/\tau_0) \ln(t/\tau_1)\right] . \tag{8}$$

For $|\omega| \gtrsim \Omega_2 = \tau_0^{-1} \exp[-1/r_0]$, this expression reduces to equation (7). Note that equation (8) amounts to an exponentiation of the perturbative result in the time domain, whereas in equation (7) the perturbation series is exponentiated in frequency space. We will further discuss equation (8) in Section 3.

The zero bias anomaly in $1d$ has so far received much less attention than the corresponding anomaly in $2d$. Recent experiments on metallic carbon nanotubes have motivated Mishchenko *et al.* [12] to study the fate of the perturbative Altshuler-Aronov correction in $1d$ at low frequencies. They found that long-range Coulomb interactions in a quasi $1d$ metal lead for $\omega \to 0$ to an exponentially small DOS,

$$\nu(\omega) \propto \exp\left[-\frac{\epsilon_0}{|\omega|} \ln\left(\frac{\epsilon_1}{|\omega|}\right)\right] , \tag{9}$$

where ϵ_0 and ϵ_1 are some finite energy scales. We shall further comment on this result below. A similar result was also obtained by Rollbühler and Grabert [13]. For frequencies exceeding a crossover scale $D_0/(2\pi R)^2$, where R is the radius of the nanotube, Egger and Gogolin [14] found a crossover to two-dimensionality, which results in a power-law dependence of the DOS. However, below this crossover scale, $1d$ behavior can be expected.

The rest of this work is organized as follows: In Section 2 we critically review the non-perturbative method leading to the above results for the zero bias anomaly. In particular, we argue that only in the case of long-range Coulomb interactions in $2d$ the method can be formally justified. In Section 3 we generalize the calculation of reference [10] to finite temperatures and present numerical results for the frequency-dependence of the average DOS which in principle can be compared with experimentally measured tunneling conductances. In Section 4 we discuss the DOS in quasi $1d$. Finally, we end with a brief summary.

2 Summing the leading singularities via a gauge transformation

The average DOS of a d-dimensional interacting Fermi system with volume V can be written at finite temperature $T = 1/\beta$ in terms of the disorder-averaged Green function at coinciding space points $\mathcal{G}(\omega) \equiv \overline{\mathcal{G}(\mathbf{r}, \mathbf{r}, \omega)}$ as

$$\nu(\omega, T) = -\frac{1}{\pi} \coth\left(\frac{\beta\omega}{2}\right) \mathrm{Im}\, \mathcal{G}(\omega) \,. \tag{10}$$

To make contact with the disorder-averaged imaginary-time Green function at coinciding space points

$G(\tau) \equiv \overline{G(\mathbf{r},\mathbf{r},\tau)}$, we notice that due to particle-hole symmetry near the Fermi energy we have

$$\mathcal{G}(\omega) = -2\int_0^\infty \mathrm{d}t\, \sin\omega t\; G(\tau \to \mathrm{i}t + 0^+)\;. \tag{11}$$

The problem of calculating the average DOS is now reduced to the problem of calculating the disorder-averaged imaginary-time Green function.

An attempt to calculate $\nu(\omega)$ within perturbation theory fails at low frequencies because the perturbative expansion of $\nu(\omega)$ is plagued by singularities, see equations (1) and (4). In $2d$ with long-range Coulomb interactions, these singularities diverge as $\ln^2\omega$. At the same time, however, the perturbative expansion of the conductivity $\sigma(\omega)$ contains only less severe $\ln\omega$-singularities [1,7]. As a consequence, for sufficiently small frequencies, we may re-sum the most singular terms in the expansion of the average DOS *without considering simultaneously a similar re-summation for the average conductivity*, because we know *a priori* that $\ln^2$-singularities do not appear in the calculation of $\sigma(\omega)$. In this sense, the problem of calculating the average DOS decouples from the problem of calculating the average conductivity. Under these conditions we may sum the most singular contributions to the average DOS *via* a simple gauge transformation: For a given realization of the disorder $U(\mathbf{r}) = \frac{1}{V}\sum_{\mathbf{q}} \mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\mathbf{r}} U_{\mathbf{q}}$, the imaginary-time Green function can be written in real space in terms of path integrals as

$$G(\mathbf{r},\mathbf{r}';\tau) \equiv -\left\langle \psi(\mathbf{r},\tau)\bar{\psi}(\mathbf{r}',0)\right\rangle\;, \tag{12}$$

where the integration measure $\mathcal{D}\{\psi\}$ in

$$\langle \ldots \rangle = \int \mathcal{D}\{\psi\} \ldots \mathrm{e}^{-S\{\psi\}} \tag{13}$$

is assumed to be properly normalized and the action $S\{\psi\}$ is given by

$$\begin{aligned} S\{\psi\} &= \int_K \left[-\mathrm{i}\tilde{\omega}_n + \mathbf{k}^2/2m - \mu\right]\bar{\psi}_K\psi_K \\ &\quad + \int_K\int_Q U_{\mathbf{q}}\bar{\psi}_{K+Q}\psi_K + \frac{1}{2}\int_Q f_{\mathbf{q}}\,\rho_{-Q}\rho_Q\;. \end{aligned} \tag{14}$$

Here we have used the collective label $K = (\mathbf{k}, \mathrm{i}\tilde{\omega}_n)$ for momentum $\mathbf{k}$ and fermionic Matsubara frequency $\tilde{\omega}_n = (2n+1)\pi/\beta$, and $Q = (\mathbf{q}, \mathrm{i}\omega_m)$ is the collective label for momentum $\mathbf{q}$ and bosonic Matsubara frequency $\omega_m = 2\pi m/\beta$.

$$\psi_K = \int_V \mathrm{d}^d r \int_0^\beta \mathrm{d}\tau\, \mathrm{e}^{-\mathrm{i}(\mathbf{k}\cdot\mathbf{r} - \tilde{\omega}_n\tau)}\psi(\mathbf{r},\tau)\;, \tag{15}$$

$$\bar{\psi}_K = \int_V \mathrm{d}^d r \int_0^\beta \mathrm{d}\tau\, \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{r} - \tilde{\omega}_n\tau)}\bar{\psi}(\mathbf{r},\tau) \tag{16}$$

are conjugate (fermionic) Grassmann fields and

$$\rho_Q = \int_K \bar{\psi}_K\psi_{K+Q} \tag{17}$$

is the Grassmann representation of the density operator. $\int_K$ is short for $\frac{1}{\beta V}\sum_{\mathbf{k},\tilde{\omega}_n}$ and $\int_Q$ is an abbreviation of $\frac{1}{\beta V}\sum_{\mathbf{q},\omega_m}$. In the following we will also use $\delta_Q \equiv \beta V \delta_{\mathbf{q},0}\,\delta_{\omega_m,0}$.

The electron-electron interaction may be decoupled with the help of a Hubbard-Stratonovich transformation: Using the identity

$$\mathrm{e}^{-\frac{1}{2}\int_Q f_{\mathbf{q}}\,\rho_{-Q}\rho_Q} = \int \mathcal{D}\{\phi\}\mathrm{e}^{-\mathrm{i}\int_Q \rho_{-Q}\,\phi_Q}\mathrm{e}^{-\frac{1}{2}\int_Q f_{\mathbf{q}}^{-1}\,\phi_{-Q}\phi_Q}, \tag{18}$$

we get an action quadratic in the Grassmann fields:

$$\begin{aligned} S\{\psi,\phi\} = \int_K\int_Q \left[-\left(G_0^{-1}\right)_{K+Q,K} + \mathrm{i}\phi_Q\right]\bar{\psi}_{K+Q}\psi_K \\ + \frac{1}{2}\int_Q f_{\mathbf{q}}^{-1}\,\phi_{-Q}\phi_Q\;. \end{aligned} \tag{19}$$

$\left(G_0^{-1}\right)_{K+Q,K} = \left[\mathrm{i}\tilde{\omega}_n - \mathbf{k}^2/2m + \mu\right]\,\delta_Q - U_{\mathbf{q}}$ is the inverse of the non-interacting Matsubara Green function for the given realization of disorder. If $G(\mathbf{r},\mathbf{r}';\tau;\phi)$ satisfies

$$\begin{aligned} [-\partial_\tau + \nabla_{\mathbf{r}}^2/2m + \mu - \mathrm{i}\phi(\mathbf{r};\tau) - U(\mathbf{r})]\,G(\mathbf{r},\mathbf{r}';\tau;\phi) = \\ \delta(\mathbf{r}-\mathbf{r}')\delta^*(\tau)\;, \end{aligned} \tag{20}$$

where $\phi(\mathbf{r},\tau) = \int_Q \mathrm{e}^{\mathrm{i}(\mathbf{r}\cdot\mathbf{q} - \omega_m\tau)}\phi_Q$ and $\delta^*(\tau) \equiv \sum_{\tilde{\omega}_n}\mathrm{e}^{\mathrm{i}\tilde{\omega}_n\tau}$ is an antiperiodic delta-function, we may easily integrate out the fermionic fields and write $G(\mathbf{r},\mathbf{r}';\tau)$ as

$$G(\mathbf{r},\mathbf{r}';\tau) = \int \mathcal{D}\{\phi\}\,G(\mathbf{r},\mathbf{r}';\tau;\phi)\,\mathrm{e}^{-S\{\phi\}}\;. \tag{21}$$

The action $S\{\phi\}$ only depends on the bosonic Hubbard-Stratonovich field ϕ and is given by

$$S\{\phi\} = \frac{1}{2}\int_Q f_{\mathbf{q}}^{-1}\,\phi_{-Q}\phi_Q - \mathrm{Tr}\,\ln\left[1 - \mathrm{i}\hat{G}_0\hat{\phi}\right]\;. \tag{22}$$

Both $\hat{G}_0$ and $\hat{\phi}$ are infinite matrices in momentum and frequency space with matrix elements $(G_0)_{K,K'}$ and $\phi_{K-K'}$, respectively. Expanding the trace of the logarithm and retaining only the Gaussian term, we obtain

$$S\{\phi\} = \frac{1}{2}\int_Q\int_{Q'} (f_{\mathrm{RPA}}^{-1})_{Q,Q'}\,\phi_{-Q}\phi_{Q'} \tag{23}$$

where for the given realization of disorder

$$(f_{\mathrm{RPA}}^{-1})_{Q,Q'} = f_{\mathbf{q}}^{-1}\,\delta_{Q-Q'} + (\Pi_0)_{Q,Q'} \tag{24}$$

is the dynamically screened interaction in the random phase approximation (RPA) and

$$(\Pi_0)_{Q,Q'} = -\int_K\int_{K'} (G_0)_{K+Q,K'+Q'}(G_0)_{K',K} \tag{25}$$

is the polarization. It should be noted that for a clean system with a linearized energy dispersion and dominant

forward scattering this Gaussian approximation becomes exact [15]. Since the $\ln^2$-singularity is due to forward scattering and the disorder is assumed to be weak the Gaussian approximation made here seems reasonable. We will come back to this approximation in Section 3.

From perturbation theory [1] we know that the $\ln^2$-singularity of the DOS is due to momenta and frequencies satisfying

$$|\omega_m|/D_0\kappa \ll |\mathbf{q}| \ll (|\omega_m|/D_0)^{1/2} \ , \quad |\omega_m| \ll \tau_0^{-1} \ . \quad (26)$$

Let us therefore separate the Hubbard-Stratonovich field into a part involving only these dangerous modes, ϕ_d, and a harmless part $\phi_h \equiv \phi - \phi_d$. Indicating summation over momenta and frequencies satisfying equation (26) by a prime, the field ϕ_d can be written as

$$\phi_d(\mathbf{r},\tau) = \int_Q{}' \, \mathrm{e}^{\mathrm{i}(\mathbf{q}\cdot\mathbf{r}-\omega_m\tau)}\phi_Q \ . \quad (27)$$

To sum up the leading divergences *via* a gauge transformation, we make the ansatz

$$G(\mathbf{r},\mathbf{r}';\tau;\phi) = G_1(\mathbf{r},\mathbf{r}';\tau;\phi)\mathrm{e}^{\mathrm{i}(\Phi(\mathbf{r};\tau)-\Phi(\mathbf{r}';0))} \quad (28)$$

and choose $\Phi(\mathbf{r};\tau)$ such that $\partial_\tau\Phi(\mathbf{r};\tau) = \phi_d(\mathbf{r};\tau)$. In this case, $G_1(\mathbf{r},\mathbf{r}';\tau;\phi)$ satisfies the differential equation

$$[-\partial_\tau - (-\mathrm{i}\nabla_\mathbf{r} - \mathbf{A}(\mathbf{r};\tau))^2/2m + \mu - \mathrm{i}\phi_h(\mathbf{r};\tau) - U(\mathbf{r})] \times G_1(\mathbf{r},\mathbf{r}';\tau;\phi) = \delta(\mathbf{r}-\mathbf{r}')\delta^*(\tau) \ , \quad (29)$$

where $\mathbf{A}(\mathbf{r};\tau) = -\nabla_\mathbf{r}\Phi(\mathbf{r};\tau)$ is a longitudinal vector potential. The above gauge transformation is familiar from classical electrodynamics where it is also possible to gauge an electric field $\mathbf{E} = -\nabla\phi$ such that $\mathbf{E} = -\partial_t\mathbf{A}$. Instead of the dangerously singular charge vertices giving rise to the $\ln^2$-singularity, the current vertices associated with the longitudinal vector potential $\mathbf{A}$ can only lead to a ln-singularity of G_1. Since we are only interested in the most singular corrections to the DOS, we may approximate

$$G_1(\mathbf{r},\mathbf{r}';\tau;\phi) \approx G_0(\mathbf{r},\mathbf{r}';\tau) \ , \quad (30)$$

where G_0 is the Green function of free fermions for the given disorder potential.

Solving $\partial_\tau\Phi(\mathbf{r};\tau) = \phi_d(\mathbf{r};\tau)$ for $\Phi(\mathbf{r};\tau)$, we obtain

$$\Phi(\mathbf{r};\tau) = \int_Q{}' \, \frac{\mathrm{i}\phi_Q}{\omega_m}\mathrm{e}^{\mathrm{i}(\mathbf{q}\cdot\mathbf{r}-\omega_m\tau)} \ . \quad (31)$$

The integral over the Hubbard-Stratonovich field may now be done resulting in

$$G(\mathbf{r},\mathbf{r}';\tau) \approx G_0(\mathbf{r},\mathbf{r}';\tau)\mathrm{e}^{Q(\mathbf{r},\mathbf{r}';\tau)} \ . \quad (32)$$

Here $G_0(\mathbf{r},\mathbf{r}';\tau)$ is the Green function of free fermions and the Debye-Waller factor $Q(\mathbf{r},\mathbf{r}';\tau)$ is given by

$$Q(\mathbf{r},\mathbf{r}';\tau) = -\int_Q\int_{Q'} \frac{(f_{\mathrm{RPA}})_{Q,Q'}}{\omega_m^2} \times \left[\mathrm{e}^{-\mathrm{i}(\mathbf{q}\cdot\mathbf{r}-\omega_m\tau)} - \mathrm{e}^{-\mathrm{i}\mathbf{q}\cdot\mathbf{r}'}\right]\left[\mathrm{e}^{\mathrm{i}(\mathbf{q}'\cdot\mathbf{r}-\omega_m\tau)} - \mathrm{e}^{\mathrm{i}\mathbf{q}'\cdot\mathbf{r}'}\right] . \quad (33)$$

To calculate the DOS, it is sufficient to know the disorder-average of $G(0,0;\tau) \approx G_0(0,0;\tau)\exp(Q(0,0;\tau))$. Following Finkelstein [7] we factorize the disorder average $\overline{G(0,0;\tau)}$ as $\overline{G(0,0;\tau)} \approx \overline{G_0(0,0;\tau)}\exp(\overline{Q(0,0;\tau)})$. With this factorization, the imaginary time Green function at finite temperature T can be written as

$$G(\tau) \approx G_0(\tau)\mathrm{e}^{Q(\tau)} \ , \quad (34)$$

where $G_0(\tau) \equiv \overline{G_0(0,0;\tau)}$ is the disorder-averaged Green function of free fermions,

$$G_0(\tau) = -\nu_0\,\frac{\pi/\beta}{\sin(\pi\tau/\beta)} \ , \quad (35)$$

and the Debye-Waller factor $Q(\tau) \equiv \overline{Q(0,0;\tau)}$ is given by

$$Q(\tau) = -\frac{1}{\beta V}\sum_{\mathbf{q},\omega_m}{}' \, \frac{f^{\mathrm{RPA}}_{\mathbf{q},\mathrm{i}\omega_m}}{\omega_m^2}\left[1-\cos(\omega_m\tau)\right] \ . \quad (36)$$

Here, $f^{\mathrm{RPA}}_{\mathbf{q},\mathrm{i}\omega_m} \equiv \beta V\overline{(f_{\mathrm{RPA}})_{Q,Q}}$ is the dynamically screened averaged Coulomb interaction. Diagrammatically, the above factorization of the disorder amounts to neglecting all diagrams involving disorder lines which do not dress the bare propagator or give a diffusion contribution. In the language of the replica sigma model used in references [7,8] this approximation should be equivalent to including only soft modes. The diagrams actually contributing to equations (34–36) are shown in Figure 1. It is important to emphasize that in deriving equations (34–36) the sub-leading logarithmic corrections have been ignored, so that it would be inconsistent to retain sub-leading terms involving only a single logarithm in the evaluation of the Debye-Waller factor.

Apparently, the above gauge-transformation method for re-summing the leading terms in the perturbative expansion of the single-particle Green function has first been used by Nazarov [16]. Later, several authors employed this technique to calculate the DOS of disordered interacting electrons [9–11]. However, one should keep in mind that in practice this method relies on the fact that the problem of calculating the average single-particle Green function decouples in the sense discussed above from the problem of calculating the conductivity. In particular, for $2d$ disordered electrons subject to *short-range interactions*, the perturbative calculation of $\nu(\omega)$ and $\sigma(\omega)$ both involve $\ln\omega$-singularities, so that a naive application of the above gauge-transformation trick for short-range interactions in $2d$ is at least problematic. This is also the case for diffusive quasi $1d$ electrons, where the perturbative calculation of both $\nu(\omega)$ and $\sigma(\omega)$ leads to $|\omega|^{-1/2}$-singularities [1]. Hence, also in this case it is problematic to calculate the average DOS using equations (34–36) without considering simultaneously the low-frequency behavior of the conductivity. We shall come back to this point in Section 4.

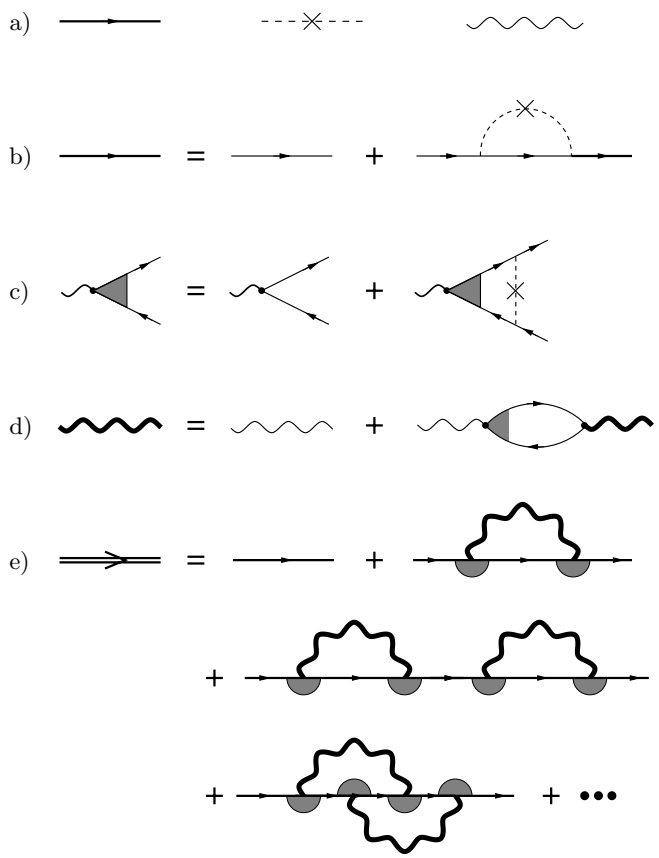

Fig. 1. Feynman diagrams contributing to the full Green function $G(\tau)$ in the approximation discussed in the text: a) diagrammatic representation of the bare Green function, the disorder line, and the bare interaction; b) disorder averaged Green function in the Born approximation; c) diffusion renormalized vertex; d) dynamically screened RPA interaction; e) diagrammatic representation of the full Green function $G(\tau)$.

3 Zero bias anomaly in 2d

For the frequency-momentum regime defined in equation (26) the RPA-interaction is given in $2d$ by [1]

$$f^{\rm RPA}_{\mathbf{q},{\rm i}\omega_m} \approx (2D_0\nu_0)^{-1}|\omega_m|/\mathbf{q}^2. \tag{37}$$

Converting the sum over Matsubara frequencies into an integral and performing the momentum integral in the thermodynamic limit $V \to \infty$, the Debye-Waller factor (analytically continued to real time) may be written as

$$Q({\rm i}t) = \frac{r_0}{2}\int_0^{\tau_0^{-1}} \frac{{\rm d}\omega}{\omega}\,\ln\left(\frac{\omega}{D_0\kappa^2}\right) \times\left[\frac{1-\cos(\omega t)}{\tanh(\beta\omega/2)} + {\rm i}\sin(\omega t)\right]. \tag{38}$$

To be consistent with the approximation made in equation (34), we retain only $\ln^2$-singularities and ignore all terms involving only single logarithms: Within this approximation the imaginary part of $Q({\rm i}t)$ vanishes, and for arbitrary temperatures we find

$$Q({\rm i}t) \sim -\frac{r_0}{4}\ln(t/\tau_1)\ln(t/\tau_0). \tag{39}$$

The DOS at finite temperature $T = 1/\beta$ is now given by

$$\nu(\omega,T) \approx \nu_0 \coth\left(\frac{\beta\omega}{2}\right)\frac{2}{\beta}\int_{\tau_0}^{\infty}{\rm d}t\,\frac{\sin(\omega t)}{\sinh(\pi t/\beta)} \times\exp\left[-\frac{r_0}{4}\ln(t/\tau_1)\ln(t/\tau_0)\right]. \tag{40}$$

For $T = 0$, equation (40) reduces to equation (8) which for $|\omega| \gg \Omega_2$ reduces to equation (7). If we consider the DOS at the Fermi energy as a function of temperature, we find for $T \gg \Omega_2$ an equation similar to equation (7), with ω replaced by $2T$. Hence, for $|\omega|$ or $2T \gg \Omega_2$, we have

$$\nu(\omega,T) \approx \nu_0\exp\Big[-\frac{r_0}{4}\ln(\max(|\omega|,2T)\tau_0) \times\ln(\max(|\omega|,2T)\tau_1)\Big]. \tag{41}$$

Setting $T = 0$ and then taking $\omega \to 0$, the leading term in the asymptotic expansion of equation (40) is

$$\nu(\omega) \sim \nu_0\frac{4}{\pi^{1/2}}\left(\frac{\tau_1}{\tau_0 r_0}\right)^{1/2}\frac{|\omega|}{\Omega_2}. \tag{42}$$

Noting that $\sqrt{\tau_0\tau_1} = r_0/(\nu_0 e^4)$, equation (42) can also be written as

$$\nu(\omega) \sim C\frac{|\omega|}{e^4}, \tag{43}$$

where the numerical constant C is given by

$$C = 4(r_0/\pi)^{1/2}\exp[1/r_0]. \tag{44}$$

On the other hand, if we first set $\omega = 0$ and then consider the leading behavior at low temperatures, we obtain for $T \ll \Omega_2$

$$\nu(0,T) \sim 2C\frac{T}{e^4}. \tag{45}$$

Thus, at low temperatures the average DOS at the Fermi energy vanishes linearly in T. For $|\omega|, 2T \ll \Omega_2$, equations (43, 45) turn into

$$\nu(\omega,T) \sim C\,\frac{\max(2T,|\omega|)}{e^4}. \tag{46}$$

A plot of the DOS for various temperatures is shown in Figure 2.

Surprisingly, equation (46) resembles the well-known classical Efros-Shklovskii Coulomb gap of two-dimensional electrons in the localized regime, where the DC conductivity $\sigma(0)$ vanishes [17]. Note, however, that in the derivation of equation (46) we have assumed that the DC conductivity $\sigma(0)$ remains finite, *i.e.* the electrons are

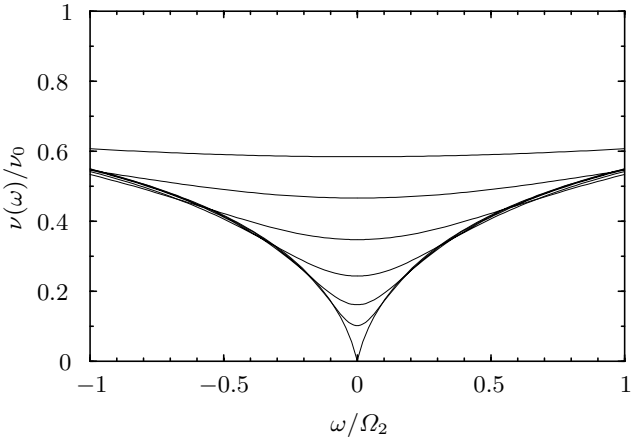

Fig. 2. Graph of the average DOS $\nu(\omega)$ in $2d$ for various temperatures, see equation (40). We have chosen $r_0 \to r_* = 1/\pi$ and $\tau_1 = \tau_0$. Curves from top to bottom are for $T/\Omega_2 = 0.8, 0.4, 0.2, 0.1, 0.05, 0.025, 0$.

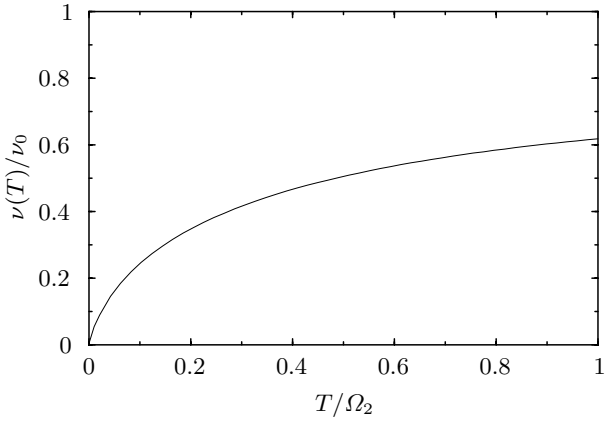

Fig. 3. Temperature-dependence of the average DOS $\nu(T)$ at the Fermi energy in $2d$. The parameters are again $r_0 \to r_* = 1/\pi$ and $\tau_1 = \tau_0$.

assumed to be *delocalized*. The difference between the localized and the delocalized regime manifests itself in the dimensionless prefactor C: whereas in the case of the classical Coulomb gap of localized electrons the constant C is a number of the order of unity that depends on the geometry of the underlying lattice, for the quantum Coulomb gap discussed here, C depends on the dimensionless conductivity of the system. The existence of the Coulomb gap in the delocalized regime of a disordered interacting $2d$ metallic system was also found numerically by Efros and Pikus [18]. More recently, an intermediate delocalized phase in small clusters of disordered interacting electrons has been found numerically in reference [19].

Formally, the derivation of equation (40) is only valid in the limit of weak disorder where $r_0 = 1/\pi k_F \ell \ll 1$. We have implicitly assumed that the DC conductivity $\sigma(0)$ does not deviate significantly from the conductivity $\sigma(\omega)$ at frequency $\omega \approx 1/\tau_0$. However, if the conductivity has a finite DC limit $\sigma(0)$, then it is reasonable to expect that the qualitative behavior of the DOS can be obtained by simply replacing $r_0 \to r_* = (e^2/h)/\pi\sigma(0)$ in equation (40). This replacement can be justified *via* a simple renormalization group argument [10]: As already discussed in the preceding section, for a disordered system the Gaussian approximation for the Hubbard-Stratonovich field ϕ is not exact. In $2d$, the Gaussian part of the *averaged* bosonic action can approximately be written as

$$\overline{S}_2\{\phi\} = \int_Q \frac{q^2}{r_0|\omega_m|}\,\phi_{-Q}\phi_Q \ . \tag{47}$$

If we consider the limits $V \to \infty$ and $T \to 0$, we have $\int_Q = \int \frac{\mathrm{d}\omega}{2\pi} \int \frac{\mathrm{d}^2 q}{(2\pi)^2}$. It turns out that $\overline{S}_2\{\phi\}$ is invariant under the scaling transformation $\mathbf{q} \to \mathbf{q}' = b\mathbf{q}$, $\omega \to \omega' = b^2\omega$ and $\phi_Q \to \phi'_{Q'} = \phi_Q/b^2$. Here $b > 1$ is the length rescaling factor. Let us now approximate the vertices occuring in the non-Gaussian terms in an expansion of $\overline{S}\{\phi\}$ by constants, such that the nth-order term is

$$\overline{S}_n\{\phi\} = u_n \int_{Q_1} \cdots \int_{Q_n} \delta_{Q_1+\cdots+Q_n}\,\phi_{Q_1}\cdots\phi_{Q_n} \ . \tag{48}$$

Then it is easy to see that under the above transformation

$$u_n \to u'_n = b^{-2(n-2)} u_n \ . \tag{49}$$

Hence, under a usual renormalization group step of a Wilsonian RG including the elimination of modes with large momenta or high frequencies and a successive rescaling as considered above, the u_n's become smaller and r_0 gets renormalized. If we consider a metallic system with a finite conductivity $\sigma(0)$, then integrating out UV-modes leads us to a Gaussian fixed point with r_0 replaced by $r_* = (e^2/h)/\pi\sigma(0)$. Thus the infrared physics is correctly described by a Gaussian action with $r_0 \to r_*$. The connection between the low-frequency behavior of the conductivity and the DOS has also been emphasized by Nazarov [16], and by Levitov and Shytov [9].

For a possible comparison of our results with future tunneling experiments on semiconductor materials which apparently show a metal-insulator transition [5,6] (as far as we know, such experiments have not been performed yet), we have plotted in Figure 2 the frequency-dependence predicted by equation (40) for $r_0 \to r_* = 1/\pi$, corresponding to the zero-temperature conductivity of order $\sigma(0) \approx e^2/h$. Saturation values in this regime are typically encountered in the metallic regime close to the apparent metal-insulator transition [6]. The temperature-dependence of the average DOS at the Fermi energy is shown in Figure 3.

It should be noted that in deriving equation (40) we have consistently neglected corrections involving single logarithms. We cannot exclude the possibility that these corrections lead to a further depletion of the Coulomb gap. For example, taking the (sub-leading) imaginary part of the full Debye-Waller factor given in equation (38) into account would lead to a nonlinear suppression of the DOS. However, as discussed above, such an approximation would not be systematic.

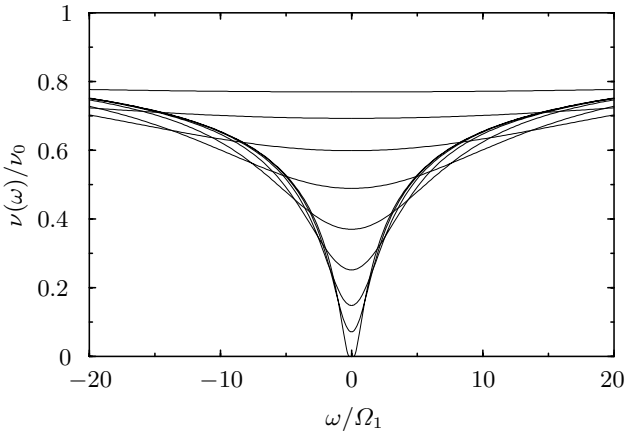

Fig. 4. Graph of the average DOS $\nu(\omega)$ in $1d$ for various temperatures, see equation (51). The curves from top to bottom are for $T/\Omega_1 = 32, 16, 8, 4, 2, 1, 0.5, 0.25, 0$.

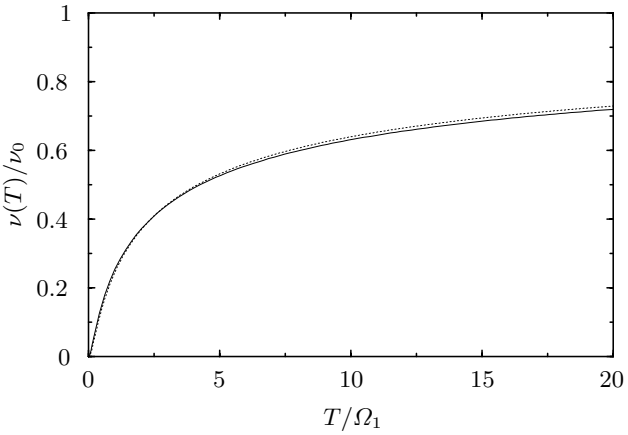

Fig. 5. Solid line: graph of the average DOS $\nu(T)$ for $\omega = 0$, see equation (51). Dotted dotted line: the approximation $\exp(-\sqrt{2\Omega_1/T})$.

4 Zero bias anomaly in 1d

As discussed in Section 2, the naive application of equations (34–36) in quasi $1d$ is problematic, because in this case the interaction corrections to the DOS and to the conductivity both involve the same type of singularities. Keeping this caveat in mind, let us nevertheless briefly discuss the predictions of equations (34–36) in quasi $1d$. A similar calculation has recently been performed in reference [12].

Since screening is much less effective in one dimension than in higher dimensions, we simply approximate the one-dimensional RPA-interaction by a constant, $f^{\rm RPA}_{q,{\rm i}\omega_m} \approx f_0$ [1]. This approximation should be correct up to logarithmic corrections in frequency. A calculation analogous to the one leading to equation (38) results in

$$Q({\rm i}t) = -\sqrt{2\Omega_1}\left[\int_0^\infty \frac{{\rm d}\omega}{\sqrt{2\pi}}\,\frac{1-\cos(\omega t)}{\omega^{3/2}\tanh(\beta\omega/2)} + {\rm i}\sqrt{t}\right], \tag{50}$$

where Ω_1 is given in equation (5). Note that in $1d$, there is no need for an ultraviolet cutoff. At $T = 0$, we have ${\rm Re}\,Q({\rm i}t) = {\rm Im}\,Q({\rm i}t)$, such that the imaginary part of $Q({\rm i}t)$ cannot be neglected. The DOS at finite temperature $T = 1/\beta$ can now be written as

$$\nu(\omega,T) \approx \nu_0 \coth\left(\frac{\beta\omega}{2}\right)\frac{2}{\beta}\int_0^\infty {\rm d}t\,\frac{\sin(\omega t)\cos(\sqrt{2\Omega_1 t})}{\sinh(\pi t/\beta)} \times \exp\left[-\sqrt{\frac{\Omega_1}{\pi}}\int_0^\infty {\rm d}\omega'\,\frac{1-\cos(\omega' t)}{(\omega')^{3/2}\tanh(\beta\omega'/2)}\right]. \tag{51}$$

As can be easily checked, for $|\omega| \gg \max\{\Omega_1, T\}$, equation (51) reduces to the perturbative result given in equation (4). A graph of $\nu(\omega,T)$ for different temperatures is shown in Figure 4. The DOS at the Fermi energy as a function of temperature is shown in Figure 5. The DOS is well approximated by an exponentiation of the perturbative result, $\nu(T) \approx \nu_0 \exp(-\sqrt{2\Omega_1/T})$, which is quite similar to a result found in reference [12]. At zero temperature, equation (51) may be simplified to

$$\nu(\omega, T=0) = \frac{\nu_0}{\pi}{\rm Re}\int_0^\infty {\rm d}x\,\frac{\sin x^2}{x}\exp\left[-2x\sqrt{\frac{{\rm i}\Omega_1}{|\omega|}}\right]. \tag{52}$$

For $|\omega| \ll \Omega_1$, this integral is easily evaluated within a saddle-point approximation,

$$\nu(\omega, T=0) \sim \nu_0\sqrt{\frac{|\omega|}{\pi\Omega_1}}\,\exp\left[-\frac{\Omega_1}{|\omega|}\right], \quad |\omega| \ll \Omega_1\,, \tag{53}$$

which differs from the corresponding expression given by Mishchenko *et al.* [12] (see Eq. (9)) by a different prefactor but agrees with a recent result found by Rollbühler and Grabert [13]. The exponential suppression of the DOS is quite surprising and is possibly an artifact of the inconsistent exponentiation of the $|\omega|^{-1/2}$-singularities inherent in equations (53, 9) for quasi $1d$ systems. While in $2d$ it was possible to separate leading $\ln^2$-singularities from sub-leading ln-singularities and sum up the most singular contributions, a similar separation is not possible in $1d$. The ultimate low-frequency behavior of the DOS in $1d$ might therefore be altered by diagrams neglected in equation (34).

5 Summary and conclusions

In summary, we have considered the fate of the singular perturbative corrections to the average DOS in quasi $1d$ and $2d$ disordered metals when the frequency and the temperature are reduced such that one leaves the perturbative

regime. For a $2d$ metal with long-range Coulomb interactions, it is possible to re-sum the leading $\ln^2$-singularities in the perturbative expansion of the average DOS to all orders in perturbation theory. The method relies on the fact that similar singularities do not appear in the perturbative expansion of the average conductivity. For a $2d$ system with short-range interactions or for a quasi $1d$ system a similar separation of energy scales does not exist, so that approximate expressions of the type given in equation (34), which involve the exponentiation of a certain subclass of Feynman diagrams, are problematic in this case. Thus, in practice the gauge-transformation trick described above, which has recently been employed by many authors [9–11,16], is only controlled in the case of $2d$ disordered electrons interacting with *long-range* Coulomb forces in the regime where the conductivity has a finite DC-limit. In this case this method yields a simple interpolation formula (40) for the average DOS, which predicts a smooth crossover from the perturbative regime at high frequencies to a new low-frequency regime, where $\nu(\omega, T)$ vanishes linearly in ω or T. The average DOS of *localized* classical electrons is known to show a similar frequency- or temperature-dependence [17]. In contrast, the metallic Coulomb gap discussed here has a quantum mechanical origin and requires delocalized electrons with a finite DC conductivity. Numerical evidence for such a metallic Coulomb gap has been found in reference [18].

In principle, it should be possible to verify the existence of the metallic Coulomb gap experimentally *via* tunneling experiments in strongly correlated disordered systems with a finite DC conductivity. The expected shape of a typical trace of the tunneling conductance as a function of the applied voltage is shown in Figure 2. Recent tunneling experiments by Bielejec *et al.* [2] in quench-condensed Beryllium films show a crossover from the perturbative regime with logarithmic corrections to an apparently linear Coulomb gap in the DOS. However, at the lowest temperatures the DOS exhibits a hard correlation gap, the origin of which remains an open question.

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